Aligning Solar Cookers: a case study in the use of symbolic geometry and CAS to investigate a real world problem

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Abstract
In this note, we report on two geometrical problems prompted by the analysis of solar cookers which lead to interesting applications of symbolic geometry and CAS technology. The first problem requires only a familiarity with isosceles triangles, while the second problem is a nice example of calculus optimization. Both use technology to aid the mathematical modeling process, but demand clear mathematical thinking of the modeler.

1. Introduction

Solar cookers are a promising source of local green energy in low latitude countries. By providing an economical and sustainable means to cook food and to purify water they can truthfully be said to help with global warming, world hunger and disease. They also lead to some tractable mathematical modeling problems which may be tackled with the aid of technology by high school students. We report on two such problems, which were recently addressed in an institute for teachers and students held at Saltire Software.

2. Box Solar Cookers

A simple solar cooker can be made by coating the sides and the inside of the lid of a box with reflective material, and covering the box itself with glass (fig 1) [1].

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Figure 1: A simple box solar cooker

A nice geometrical question is this: If the box sits on a flat surface, and the sun is at angle θ to horizontal, what angle should the lid be opened to?

A model of the situation in Geometry Expressions [2] is given in fig 2:

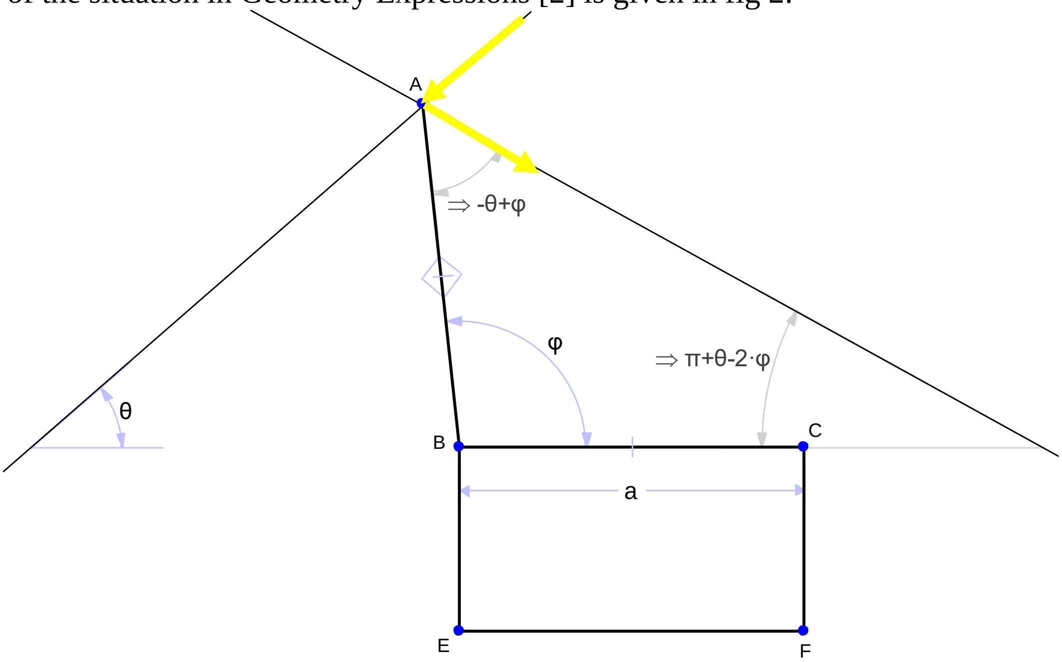

Figure 2: Geometry Expressions model of a box solar cooker. The lid of the box is positioned at angle φ to the horizontal. A ray of light at angle θ to horizontal is reflected off the tip of the lid. Geometry Expressions computes other angles in the diagram.

BC models the top of the box, and AB the lid. A ray at angle θ to BC is reflected in AB at A. A little experimentation with the model suggests that the ideal angle for the lid will put the reflected ray through point C, with the effect that all the light reflected from the lid actually hits the box. As

AB and BC are equal lengths, this would make an isosceles triangle ABC. Equating the two angles in the above diagram and solving for φ yields:

$$\varphi = \frac{2}{3}\theta + \frac{\pi}{3}$$

Now assume the box has a second flap CD reflecting light, where |CD| = |BC| (fig 3). What angle should BCD be set to so that all the reflected light hits the box?

A similar argument to the above (or substitution of π-θ for θ in the above formula) gives an angle:

$$\psi = \pi - \frac{2}{3}\theta$$

As the angle between the two flaps is

$$\alpha = \varphi + \psi - \pi$$

With the flaps in the optimal location the angle between the flaps is π/3

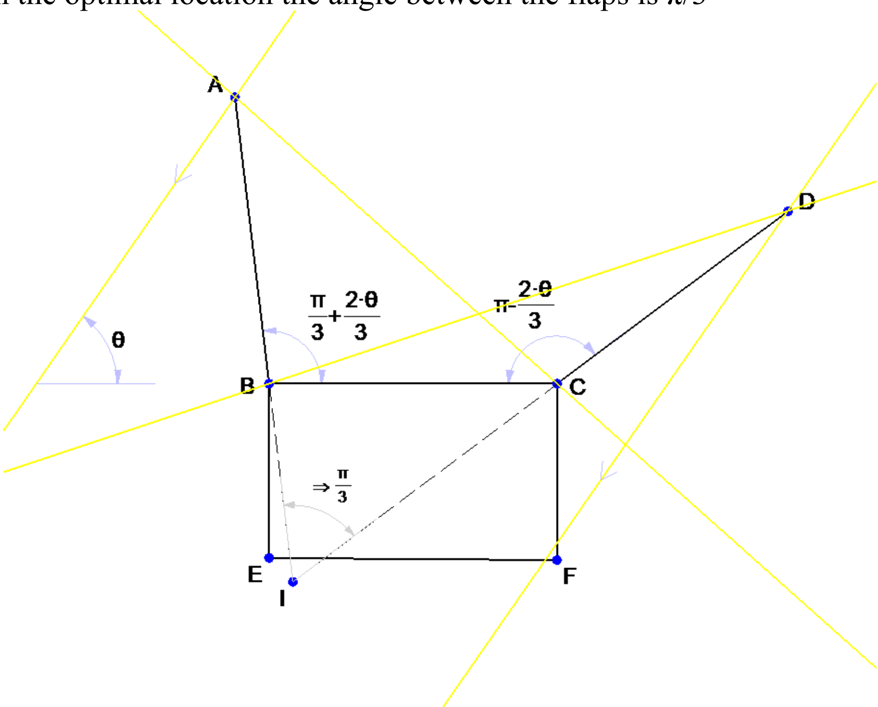

Figure 3: Model of a box solar cooker with two reflecting flaps. When optimally positioned, so that light reflected from each flap hits the box (as shown above), the flaps should be at 60 degrees to each other.

3. Parabolic Solar Cookers

Parabolic solar cookers [3] rely on the property of the parabola that light parallel to the axis is reflected to the focus. A solar cooker consists of a parabolic mirror and some apparatus to place a cooking pot at the focus of the parabola. Different designs are shown in fig 4.

Figure 4: Different solar cooker designs have different f-ratios (the ratio of focal length to diameter).

From a geometrical perspective, the difference between these designs lies in the flatness of the mirror. The flatter the mirror, the further away the focus is from the center of the mirror. The ratio of focal length to diameter of the mirror is typically referred to as the f-number. Another ratio of importance is the solar concentration ratio: the ratio of the diameter of the mirror to the diameter of the target (the cooking pot or kettle). This defines the effectiveness of the cooker. The smaller the pot size relative to the mirror size, the greater the solar concentration ratio.

The converse of the focal property of the parabola is that light rays which are not parallel to the axis miss the focus. Hence when the device is not accurately pointed at the sun, reflected light is not concentrated at the focus, and in fact misses it entirely. Now the cooking pot is finite size, so a small misalignment can be expected to still concentrate on the pot, if not on its center. Clearly the smaller the pot, the sooner we might expect to lose some of the reflected light, and the more sensitive the cooker is to misalignment. We asked ourselves the following questions:

1. For a given f number and solar concentration ratio, how much misalignment can the cooker tolerate before losing sunlight?

2. For a given solar concentration ratio, what f-number makes the cooker most tolerant to misalignment?

At what angle does the cooker drop below 100% concentration

We used Geometry Expressions to create a symbolic geometry model of the cooker. The pot was modeled as a circle centered at the focus (fig. 5).

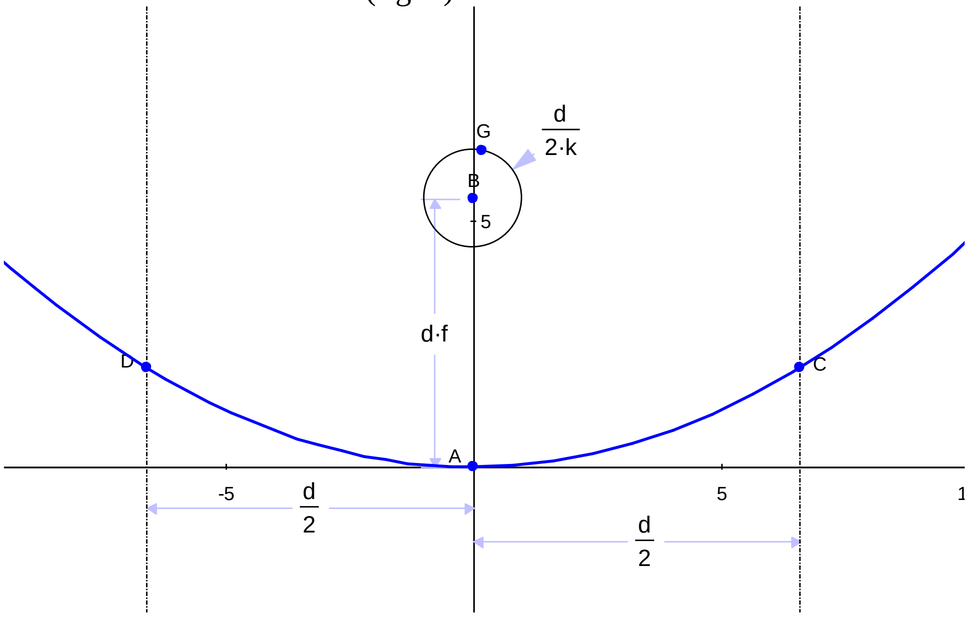

Figure 5: Geometry Expressions model of a parabolic reflector with circular target. Mirror diameter is d, f-ratio (ratio of focal length to mirror diameter) is f, and solar concentration ratio (ratio of mirror diameter to target diameter) is k.

In the model, C and D represent the edges of the mirror; d is the diameter of the mirror, f is the f number (hence the focal distance is d·f); k is the solar concentration ratio, and hence if r is the radius of the target circle, then $k = \frac{d}{2r}$ the radius of the target circle is $\frac{d}{2k}$.

First, we model a ray of light at angle θ to the axis impinging on the parabola at parametric location t. We use the symbolic geometry system to derive the distance between this ray and the focus (fig 6).

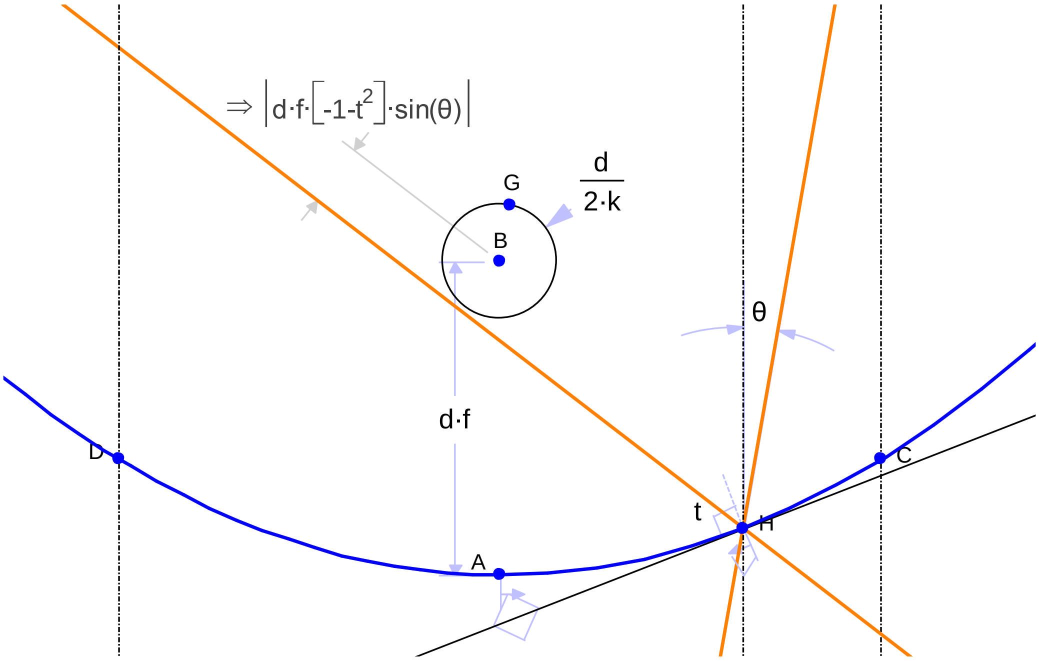

Figure 6: Distance of ray reflected from parametric location t on the parabola from the focus.

Examination of this expression shows us that the distance is symmetrical in t, and increases with increasing absolute value of t. Hence for a particular finite mirror, the light ray reflected from the edge of the mirror will be furthest from the focus. Hence to compute the angle at which a given cooker starts to lose light, we can work backwards, creating a ray which is tangent to the target circle through the edge of the mirror. The angle between the reflection of this ray and the vertical is the required angle (fig 7).

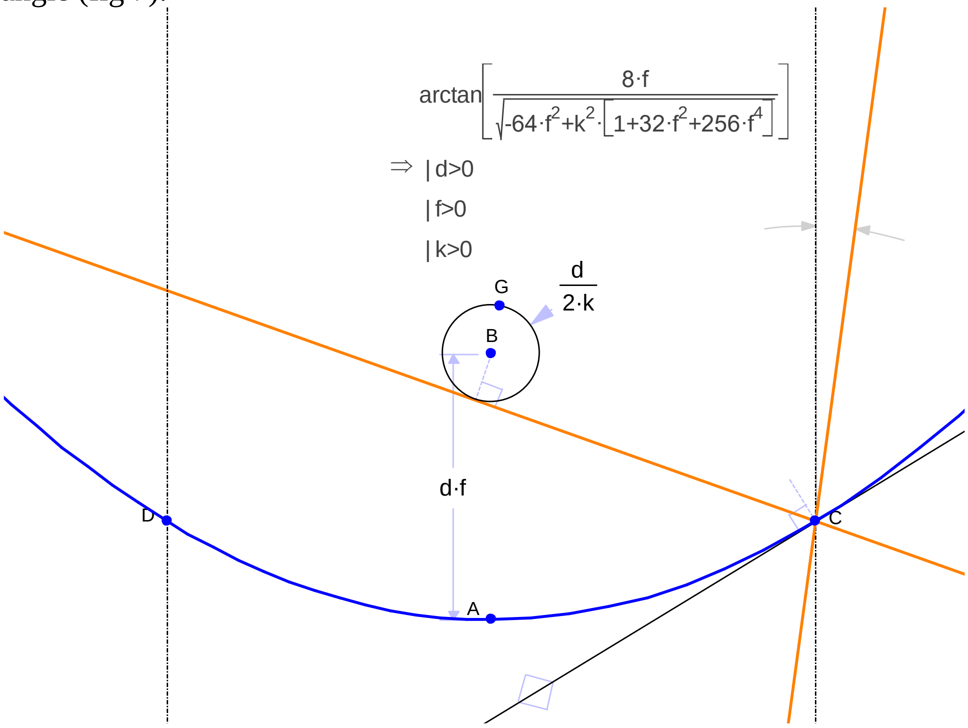

Figure 7: A ray is projected tangential to the target circle and through the edge of the parabolic mirror. The angle between its reflection and the vertical is computed by the software

Figure 8 is a graph of this angle (in degrees) for f = 0.5, and k ranging from 3 to 15

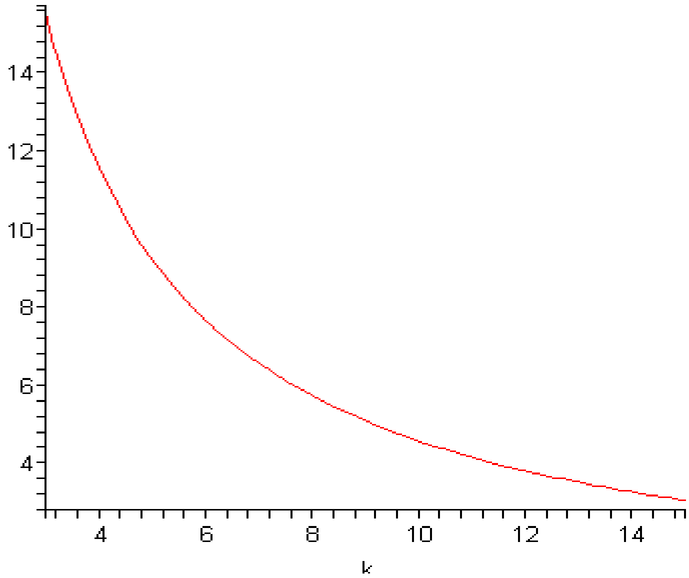

Figure 8: Critical angle in degrees as a function of solar concentration ratio for an f number of 0.5

Optimal design for tolerance to misalignment

In the graph of Figure 9, solar concentration ratio is held constant at k=8, while f varies:

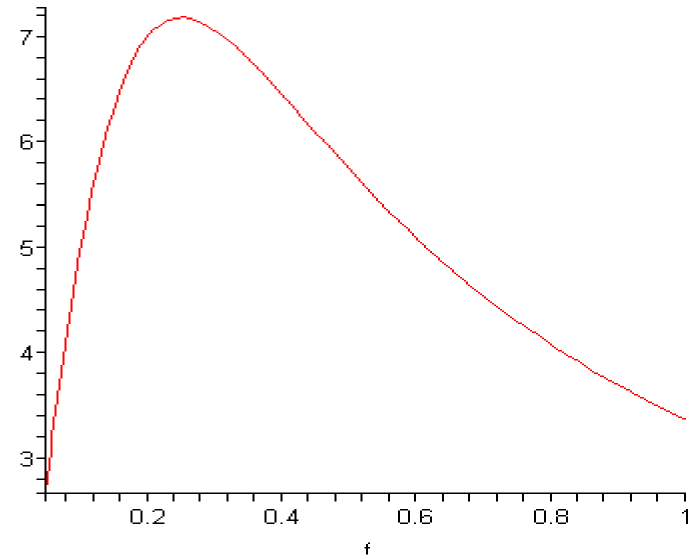

Figure 9: Critical angle as a function of f-number for a fixed solar concentration ratio of 8.

We observe a clear optimum at an f value a little over 0.2. Using Maple, we can examine this model symbolically by differentiating the expression for the critical angle and simplifying the derivative:

```
> diff(arctan(8*f/(-64*f^2+(1+32*f^2+256*f^4)*k^2)^(1/2)),f);
```

$$\frac{-\frac{4f(-128f+(64f+1024f^3)k^2)}{(-64f^2+(1+32f^2+256f^4)k^2)^{(3/2)}}+\frac{8}{\sqrt{-64f^2+(1+32f^2+256f^4)k^2}}}{1+\frac{64f^2}{-64f^2+(1+32f^2+256f^4)k^2}}$$

```
> simplify(%);
```

$$-\frac{8(16f^2-1)}{(1+16f^2)\sqrt{-64f^2+k^2+32k^2f^2+256k^2f^4}}$$

By inspection, we see that the optimum occurs when f = ¼, independent of k.
We can go back to Geometry Expressions and replace f by ¼, to examine the geometrical implications of this optimal design (fig. 10)

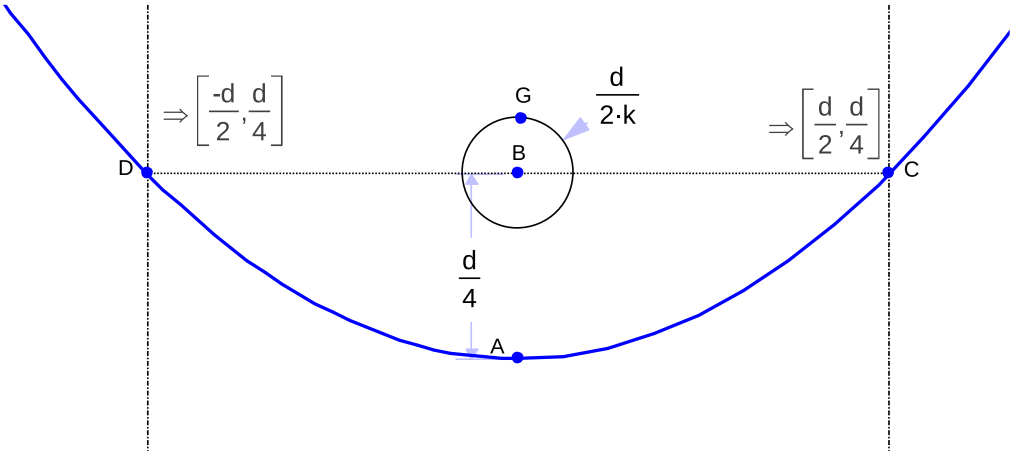

Figure 10: The optimal f-number for tolerance to misalignment is ¼. This corresponds to the situation where the focus lies on the line joining the two edges of the mirror.

We see that the greatest tolerance for misalignment occurs when the focus is directly in line with the edge of the mirror.

4. Conclusion

We have shown two mathematical modeling problems, where the use of a symbolic geometry system assisted in the modeling process, making it accessible to students at an earlier stage of their mathematical development than would otherwise be the case. Both problems were drawn from the application area of analysis of solar cooker geometry. The first problem led to a straightforward geometry problem, the second to an application of calculus.
The role of the symbolic geometry technology in both examples lay in allowing the user first to explore the problem visually, then in assisting in the conversion of this visual "feel" into a sharp mathematical statement, expressed symbolically. In the first problem the mathematical statement was that the solution to the physical problem was equivalent to the solution of a particular linear equation. In the second problem the mathematical statement involved finding the derivative of a certain function, and solving for a zero derivative. In the second case, CAS technology was useful in performing the solution steps.

Electronic Materials

Link to Geometry Expression file for figure 2 (solar cooker 1).
Link to Geometry Expression file for figure 3 (solar cooker 2).
Link to Geometry Expression file for figure 7 (solar cooker 3).

5. References

1. http://solarcooking.wikia.com/wiki/Solar_box_cookers
2. Todd, P., (2007) *Geometry Expressions: A Constraint Based Interactive Symbolic Geometry System*, Lecture Notes in Computer Science 4869, 189-202.
3. http://solarcooking.wikia.com/wiki/Parabolic_solar_reflectors